

Functions on Okounkov bodies coming from geometric valuations

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Abstract

We study topological properties of functions on Okounkov bodies as introduced by Boucksom and Chen [1], and Witt-Nyström [18]. We note that they are continuous over the whole Okounkov body whenever the body is polyhedral, on the other hand, we exhibit an example that shows that continuity along the boundary does not hold in general.

1 Introduction

We aim here to study certain functions on Okounkov bodies associated to Cartier divisors which arise from geometric valuations of the function field of the underlying variety. We will give examples of several calculations of such functions, and show how to describe them explicitly in favourable cases.

Following the pioneering work of Okounkov [16], Lazarsfeld–Mustață [15] and Kaveh–Khovanskii [10] showed how to associate a convex body to a big Cartier divisor D via studying the vanishing behaviour of global sections along a complete flag of subvarieties. This body was then called the Newton–Okounkov body of the divisor, and it soon proved to be a fundamental asymptotic invariant of D . Subsequent applications of the theory of Newton–Okounkov bodies (Okounkov bodies for short) outside complex geometry include connections to representation theory [9] and Schubert calculus [11].

Okounkov bodies can be considered as generalizations of moment polytopes in symplectic geometry; on smooth toric varieties moment polytopes are special cases of Newton–Okounkov bodies. Arguably the most interesting application of this theory so far is attached to the moment polytope point of view: in a recent paper, Harada and Kaveh [7] construct completely integrable systems on smooth projective varieties that map onto certain Okounkov bodies.

Coming from ideas in complex analytic geometry, Witt-Nyström [18] and Boucksom–Chen [1] present ways to obtain continuous functions on Okounkov bodies given a multiplicative filtration of the associated section ring. As explained by Witt-Nyström in [19], some of these functions are closely related to Donaldson’s test configurations [3, 4].

Philosophically speaking, Newton–Okounkov bodies replace the volume of a divisor $\text{vol}_X(D)$, which is just a number by a convex body, thus giving it extra structure. As a rough approximation, the value of a function associated to a valuation ν at a point of the Okounkov body is the supremum over the values of ν at sections with

the same vanishing vector. The most important property of the functions associated to filtrations is that their image measure describes the asymptotic behaviour of the jumping values of the filtration.

In this paper we consider functions arising from filtrations that carry significant geometric information, more specifically, we look at filtrations coming from geometric valuations of the function field. We work over the complex number field. Given a geometric valuation ν on an irreducible projective variety X , we obtain a filtration $\mathcal{F}_\bullet \mathbb{C}(X)$ of the function field $\mathbb{C}(X)$ by setting

$$\mathcal{F}_t \mathbb{C}(X) \stackrel{\text{def}}{=} \{f \in \mathbb{C}(X) : \nu(f) \geq t\} \quad \text{for } t \in \mathbb{R}.$$

For a big Cartier divisor D on X , this induces a multiplicative filtration on the section ring $R(X, D) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mD))$. This filtration has at most linear growth. By the method of Boucksom and Chen [1] or Witt-Nyström [18], \mathcal{F}_\bullet then gives rise to a non-negative continuous function

$$\varphi_{\mathcal{F}_\bullet} : \Delta_{Y_\bullet}(D) \longrightarrow \mathbb{R}.$$

By concavity, these functions are always continuous in the interior of the underlying Okounkov bodies, nevertheless, since continuous function on compact spaces have particularly good properties, it is important to be able to control their behaviour on the boundary. Our main result concerns exactly this question.

Theorem 1.1. *1. Let X be a projective variety, Y_\bullet an admissible flag, D a \mathbb{Q} -effective Cartier divisor on X , V_\bullet a graded linear series associated to D . Pick a geometric valuation ν of $\mathbb{C}(X)$. If the Newton–Okounkov body $\Delta_{Y_\bullet}(V_\bullet)$ is a polytope (not necessarily rational), then the Okounkov function $\varphi_\nu : \Delta_{Y_\bullet}(V_\bullet) \rightarrow \mathbb{R}$ is continuous on the whole $\Delta_{Y_\bullet}(V_\bullet)$.*

2. On the other hand, there exists a variety X , equipped with a flag $Y_1 \dots, Y_n$, and a geometric valuation on X , ν , and an ample divisor D on X such that the Okounkov function φ_ν on $\Delta(D)$ is not continuous.

The Theorem will be proven in subsections 4.1 and 4.3.

We start, in section 2, by recalling the definitions of Okounkov bodies, giving some examples of calculations, and proving some technical lemmas which will be needed in the rest of the paper. Section 3 contains definitions and technical preliminaries on filtrations of algebras. In section 4, we then present Witt-Nyström and Boucksom-Chen’s definitions of Okounkov functions. We then calculate several examples of Okounkov functions before turning to the question of invariants of Okounkov functions. We observe that given a judicious choice of a flag, the computation of $\varphi_{\mathcal{F}_\bullet}$ can be reduced to the boundary of the Okounkov body.

Theorem 1.2. *Assume that V_\bullet is a graded linear series associated to the line bundle L such that there is an irreducible divisor $Y_1 \in |L|$. We take a flag Y_\bullet whose divisorial part is Y_1 . Let \mathcal{F}_\bullet be a filtration on V_\bullet defined by a geometric valuation ν . Then for $x = (x_1, \dots, x_n) \in \Delta_{Y_\bullet}(V_\bullet)$ we have*

$$\varphi_{\mathcal{F}_\bullet}(x_1, \dots, x_n) = (1 - x_1) \varphi_{\mathcal{F}_\bullet} \left(0, \frac{x_2}{1 - x_1}, \dots, \frac{x_n}{1 - x_1} \right) + x_1 \cdot \nu(Y_1).$$

This result will be proven in Theorem 4.14 below.

In section 5, we turn to the integral of Okounkov functions. Boucksom and Chen show in passing that the integral of Okounkov functions is independent of the choice

of the flag. This gives rise to new asymptotic invariants of divisors. Note that we prove in a sequel [13] to our current paper that the maximum of an Okounkov function is independent of the chosen flag as well.

Let ν be a geometric valuation, D a Cartier divisor on X . We define

$$I(D; \nu) \stackrel{\text{def}}{=} \frac{1}{\text{vol}_X(D)} \int_{\Delta_{Y_\bullet}(D)} \varphi_\nu$$

for an arbitrary admissible flag Y_\bullet on X . Then one can interpret [1, Theorem 1.11] as saying that $I(D; \nu)$ is the limit of normalized sums of jumping values of the underlying filtration.

Theorem 1.3. *With notation as above, the invariant I_ν has the following properties.*

1. *If $D \equiv D'$, then $I_\nu(D) = I_\nu(D')$.*
2. *For a positive integer a , one has $I_\nu(aD) = a \cdot I_\nu(D)$.*
3. *There is a unique extension of I_ν to a continuous function*

$$I_\nu : \text{Big}(X) \longrightarrow \mathbb{R}_{\geq 0} .$$

The statements above will be shown in Proposition 5.5, Remark 5.7, and Proposition 5.6.

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2 Definitions and examples

2.1 Okounkov bodies

We recall here some basic notions and properties of Okounkov bodies and establish the notation. A systematic development of the theory in the geometric setting has been initiated in [15] and we refer to this article for motivation and additional details.

Let X be an irreducible projective variety of dimension n and

$$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_{n-1} \supset Y_n = \{p\}$$

be a flag of irreducible subvarieties of X such that $\text{codim}_X(Y_i) = i$ and p is a smooth point of each Y_i .

Let D be a Cartier divisor on X and let V_\bullet be a graded linear series associated to D (see [14, Definition 2.4.1]).

Remark 2.1. By a *big graded linear series* we mean one satisfying Condition (C) of [15, Definition 2.9]. Note that the definition of an Okounkov body works fine for any arbitrary graded linear series V_\bullet . In fact, it could be an interesting topic to study Okounkov bodies of non-big pseudo-effective divisors.

As explained in [15, Lemma 2.6], if V_\bullet is big, then the corresponding Okounkov body will contain an open ball.

The flag Y_\bullet defines a rank- n valuation

$$\nu_{Y_\bullet} : V_k \setminus \{0\} \rightarrow \mathbb{Z}^n$$

in the following way. Given a section $0 \neq s \in V_k \subset H^0(X, kD)$ we set

$$\nu_1 = (\nu_{Y_\bullet})_1(s) := \text{ord}_{Y_1}(s).$$

This determines a section

$$\tilde{s} \in H^0(X, kD - \nu_1 Y_1), \quad (1)$$

which does not vanish identically along Y_1 , and thus restricts to a non-zero section

$$s_1 \in H^0(Y_1, (D - \nu_1 Y_1)|_{Y_1}).$$

We repeat the above construction for s_1 and so on. In this way we produce a valuation vector

$$\nu_{Y_\bullet}(s) = ((\nu_{Y_\bullet})_1(s), \dots, (\nu_{Y_\bullet})_n(s)) \in \mathbb{Z}^n$$

and an element

$$(\nu_{Y_\bullet}(s), k) \in \Gamma_{Y_\bullet}(V_\bullet) \subset \mathbb{Z}^{n+1} \quad (2)$$

in the *graded semigroup* of the linear series V_\bullet . Let $\text{Val}_{Y_\bullet}(V_\bullet) \subset \mathbb{R}^n$ be the set of all normalized valuation vectors obtained as above i.e.

$$\text{Val}_{Y_\bullet}(V_\bullet) = \left\{ \frac{1}{k} \nu_{Y_\bullet}(s) : s \in V_k, k = 1, 2, 3, \dots \right\} \subseteq \mathbb{R}^n.$$

We write simply $\text{Val}_{Y_\bullet}(D)$ if V_\bullet is the complete linear series of D . For a given element $v \in \text{Val}_{Y_\bullet}(V_\bullet)$, we define

$$S_v \stackrel{\text{def}}{=} \{k \in \mathbb{N} \mid \exists s \in V_k : \nu_{Y_\bullet}(s) = kv\}.$$

Clearly S_v is an additive subsemigroup in \mathbb{N} .

Definition 2.2 (Okounkov body of a graded linear series). The *Okounkov body* $\Delta_{Y_\bullet}(V_\bullet)$ of V_\bullet is the closed convex hull of the set $\text{Val}_{Y_\bullet}(V_\bullet)$.

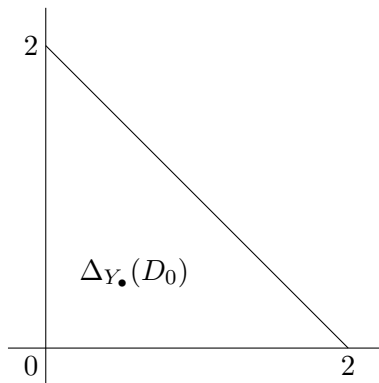
We will see below that in fact taking the closure is enough as the normalized valuation vectors are dense in the convex hull. Again, if V_\bullet is the complete linear series associated to a Cartier divisor D on X , then we write $\Delta_{Y_\bullet}(D)$ for its Okounkov body.

Remark 2.3. Note that we abuse notation slightly, since $\Delta_{Y_\bullet}(V_\bullet)$ is in general a convex compact set only. One needs V_\bullet to be big in order for the Okounkov body really contain an open ball.

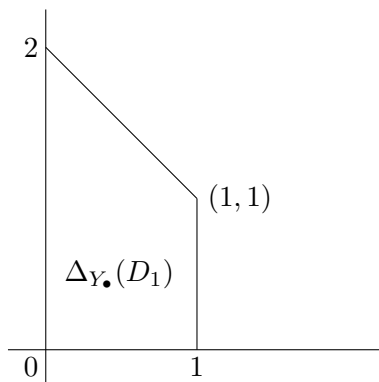
Example 2.4 (Okounkov bodies of \mathbb{P}^2 and its blow up). Let ℓ be a line in $X_0 = \mathbb{P}^2$ and $P_0 \in \ell$ a point. In what follows we operate with a fixed flag

$$Y_\bullet : X_0 \supset \ell \supset \{P_0\}.$$

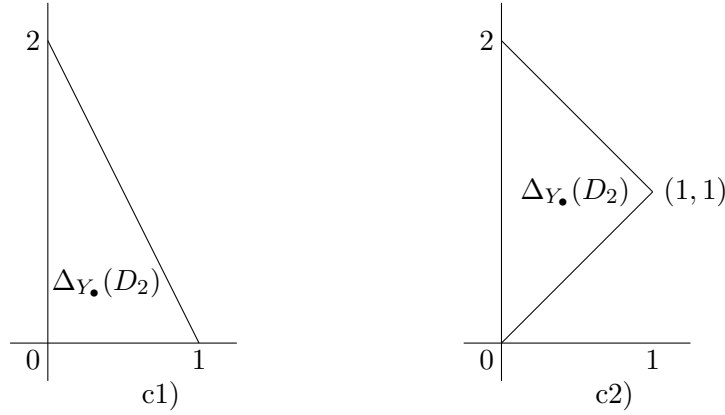
a). Let $D_0 = \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\Delta_{Y_\bullet}(D_0)$ is twice the standard simplex in \mathbb{R}^2



b). Let P_1 be a point in the plane not lying on the line ℓ and let $f_1 : X_1 = \text{Bl}_{P_1} X_0 \rightarrow X_0$ be the blow up of P_1 with exceptional divisor E_1 . For $D_1 = f_1^* \mathcal{O}_{\mathbb{P}^2}(2) - E_1$, we have



c). Let P_1, P_2 be points in the plane not lying on the line ℓ and such that P_0, P_1, P_2 are not collinear. Let $f_2 : X_2 = \text{Bl}_{P_1, P_2} X_0 \rightarrow X_0$ be the blow up of P_1, P_2 with exceptional divisors E_1, E_2 . For a big and nef line bundle $D_2 = f_2^* \mathcal{O}_{\mathbb{P}^2}(2) - E_1 - E_2$, we have then the Okounkov body as in the picture c1) below. The picture c2) shows the Okounkov body of the same line bundle under assumption that P_0, P_1, P_2 are collinear.



2.2 Density of valuation vectors

Here we verify that the points in $\text{Val}_{Y_{\bullet}}(V_{\bullet})$ are dense in the convex hull of $\text{Val}_{Y_{\bullet}}(V_{\bullet})$, hence also in $\Delta_{Y_{\bullet}}(V_{\bullet})$. This means in particular that the closure of $\text{Val}_{Y_{\bullet}}(D)$ is convex.

We first treat the case of a complete linear series $|L|$ on a curve C , because it is particularly transparent and constructive.

Fix a flag $Y_{\bullet} : C = Y_0 \supset Y_1 = \{p\}$, and recall that $\Delta_{Y_{\bullet}}(L) = [0, \deg L]$, see [15, Example 1.3]. For a given point $q \in C$ (which might or might not be equal to p), we write

$$\mathcal{S}_{v,k}(q) \stackrel{\text{def}}{=} \{t \in \mathbb{R} \mid \exists s \in V_k : \text{ord}_q(s) \geq t, \nu_{Y_{\bullet}}(s) = kv\} .$$

By definition $\mathcal{S}_{v,k}(q) \neq \emptyset$ if and only if $k \in S_v$.

Lemma 2.5 (Complete linear series on a curve). *With notation as above we have the following claims.*

1. $\text{Val}_{Y_{\bullet}}(L) \setminus \deg L = [0, \deg L) \cap \mathbb{Q}$, in particular, the set of normalized vanishing vectors is dense in $\Delta_{Y_{\bullet}}(L)$.
2. For $v \in \text{Val}_{Y_{\bullet}}(V_{\bullet})$ the set $S_v \subseteq \mathbb{N}$ is an additive subsemigroup with the exponent $e(S_v) = d$, where d is the denominator of the rational number v in its reduced form.
3. For given $v \in \text{Val}_{Y_{\bullet}}(L)$ and $q \in C$, the sequence

$$a_k \stackrel{\text{def}}{=} \frac{1}{dk} \sup \mathcal{S}_{v,dk}(q)$$

is convergent.

Remark 2.6. Let us discuss the possibility of $\deg L \in \text{Val}_{Y_{\bullet}}(L)$. By definition, this happens precisely if $H^0(C, \mathcal{O}_C(mL - m(\deg L)p)) \neq 0$ for some $m \geq 0$. This is equivalent to asking that

$$L - (\deg L)p$$

is a torsion point in $\text{Jac}(C)$. This is certainly not the case for most line bundles L on a non-rational curve C .

Proof of Lemma 2.5. (1) By construction all elements v in $\text{Val}_{Y_\bullet}(L)$ are rational numbers, and they sit inside $\Delta_{Y_\bullet}(L) = [0, \deg L]$, in particular, $v \leq \deg L$. In the other direction, let $v \in \mathbb{Q} \cap [0, \deg L]$. Let $m = kd$ be so large that

$$h^1(C, \mathcal{O}_C(mL - mv \cdot p)) = 0 \quad \text{and} \quad h^1(C, \mathcal{O}_C(mL - (mv + 1) \cdot p)) = 0. \quad (3)$$

We want to show that $m \in S_v$, i.e. that there exists a section of $\mathcal{O}_C(mL)$ vanishing at p to order exactly mv . The vanishing in (3) implies

$$h^0(C, \mathcal{O}_C(mL - mv \cdot p)) > h^0(C, \mathcal{O}_C(mL - (mv + 1) \cdot p)) \quad (4)$$

via Riemann–Roch applied on C to both systems. They are non-empty by the same token. It follows that there is a section in mL whose vanishing order at p is exactly mv . Hence $m \in S_v$.

(2) The claim that S_v is a subsemigroup of \mathbb{N} is a consequence of the fact that ν_{Y_\bullet} behaves logarithm-like on global sections. It must be $d|e(S_v)$, since mv is an integer for every $m \in S_v$. In order to show the equality, we need to check that S_v contains all natural numbers kd for $k \gg 0$.

This follows again from a Riemann–Roch computation. Let $v \in \text{Val}_{Y_\bullet}(L)$ be fixed with $v < \deg(L)$. There exists then m_0 such that for all $m \geq m_0$ one has the vanishing (3).

Let k be so that kd is an integer satisfying $kd > m_0$. Then Riemann–Roch together with the vanishing implies as above

$$h^0(C, \mathcal{O}_C(kdL - kd v \cdot p)) > h^0(C, \mathcal{O}_C(kdL - (kd v + 1) \cdot p)), \quad (5)$$

which in turn means that $kd \in S_v$.

(3) Part (2) implies that $\mathcal{S}_{v,dk}(q) \neq \emptyset$ for $k \gg 0$, hence $b_{dk} := \sup \mathcal{S}_{v,dk}(q)$ forms a super-additive sequence of rational numbers (that is, different from $-\infty$) in k . Consequently, the limit of the sequence $a_k := \frac{1}{dk} b_{dk}$ exists by [6]. \square

We now move on to the general case when the underlying variety X is allowed to have arbitrary dimension, and V_\bullet is a graded linear series.

Lemma 2.7. *Let X be a projective variety, V_\bullet a graded linear series (not necessarily big) associated to a \mathbb{Q} -effective Cartier divisor D . Then*

- (1.) *The set $\text{Val}_{Y_\bullet}(V_\bullet)$ is dense in $\Delta_{Y_\bullet}(V_\bullet)$.*
- (2.) *For $v \in \text{Val}_{Y_\bullet}(V_\bullet)$ the set $S_v \subseteq \mathbb{N}$ is an additive subsemigroup.*
- (3.) *For given $v \in \text{Val}_{Y_\bullet}(V_\bullet)$ and $q \in C$, the sequence*

$$a_k \stackrel{\text{def}}{=} \frac{1}{k} \sup \mathcal{S}_{v,k}(q)$$

with k running through the elements of S_v is convergent.

Proof. (1.) The argument now is less constructive than in the case of curves, on the other hand it explains why the closure of the set of normalized valuation vectors is a convex set. Let $v_1, v_2 \in \text{Val}_{Y_\bullet}(V_\bullet)$, and let $m_i \in \mathbb{N}$, $s_i \in V_{m_i} \subseteq H^0(X, \mathcal{O}_X(m_i D))$ for $i = 1, 2$ be such that

$$\nu_{Y_\bullet}(s_i) = m_i v_i \quad \text{for } i = 1, 2.$$

Then $s_1^{m_2} s_2^{m_1} \in V_{2m_1 m_2}$, and

$$\nu_{Y_\bullet}(s_1^{m_2} s_2^{m_1}) = m_2 \cdot \nu_{Y_\bullet}(s_1) + m_1 \cdot \nu_{Y_\bullet}(s_2) = m_2 m_1 v_1 + m_1 m_2 v_2 = m_1 m_2 (v_1 + v_2),$$

hence

$$\frac{1}{2m_1 m_2} \Gamma_{Y_\bullet}(V_{2m_1 m_2}) \ni \frac{1}{2m_1 m_2} \cdot \nu_{Y_\bullet}(s_1^{m_2} s_2^{m_1}) = \frac{1}{2}(v_1 + v_2).$$

This shows that the midpoint between two normalized valuation vectors is again a normalized valuation vector, hence density follows.

The above argument shows also that for $v_1, v_2 \in \text{Val}_{Y_\bullet}(V_\bullet)$ the segment $\overline{v_1 v_2}$ is contained in the closure $\Delta_{Y_\bullet}(V_\bullet)$, therefore the closure is a convex set.

(2.) The fact that S_v is an additive subsemigroup follows from the valuation-like behavior of ν_{Y_\bullet} and the property that $V_k \cdot V_m \subseteq V_{k+m}$.

(3.) The proof is the same as in the case of curves. \square

Remark 2.8. Note that the property (1.) in Lemma 2.7 has been silently used in the proof of [15, Proposition 2.1]. We include a proof here for the lack of a direct reference.

3 Filtrations

Filtrations of vector spaces and graded algebras are used by Boucksom and Chen [1] to define functions on Okounkov bodies. Here we recall the notions we will need, and look at the situations that are interesting from the geometric point of view. The formal considerations come from [1] for the most part.

3.1 Filtrations on vector spaces

We begin by making it precise what we mean by a filtration in this article.

Definition 3.1 (A filtration). Let E be a finite dimensional complex vector space. We call a family $\mathcal{F}_\bullet E$ of linear subspaces of E indexed by real numbers $t \in \mathbb{R}$ a *filtration* on E if

1. for all real numbers $t \in \mathbb{R}$, $\mathcal{F}_t E \subset E$ is a vector subspace;
2. \mathcal{F}_\bullet is non-increasing i.e.

$$\text{from } t_1 \leq t_2 \text{ follows } \mathcal{F}_{t_1} E \supset \mathcal{F}_{t_2} E;$$

3. \mathcal{F}_\bullet is left continuous i.e.

$$\lim_{t \rightarrow t_0^-} \mathcal{F}_t E = \mathcal{F}_{t_0} E;$$

4. \mathcal{F}_\bullet is left and right bounded i.e. there exist real numbers t_l and t_r such that

$$\mathcal{F}_{t_l} E = E \text{ and } \mathcal{F}_{t_r} E = 0.$$

A standard situation for this article is the following.

Example 3.2 (A filtration defined by a valuation). Let X be an irreducible projective variety and let $E \subset \mathbb{C}(X)$ be a finite dimensional complex vector subspace of the function field of X . Let $\nu : \mathbb{C}(X) \rightarrow \mathbb{Z}$ be a rank 1 valuation. Then

$$\mathcal{F}_t E := \{f \in E : \nu(f) \geq t\}$$

is a filtration on E .

Given a filtration we define jumping numbers.

Definition 3.3 (Jumping numbers). Let \mathcal{F}_\bullet be a filtration on a finite dimensional vector space E . The numbers

$$e_j(E, \mathcal{F}_\bullet) := \sup \{t \in \mathbb{R} : \dim \mathcal{F}_t E \geq j\}$$

for $j = 1, \dots, \dim E$ are the *jumping numbers* of the filtration \mathcal{F}_\bullet . We suppress E and \mathcal{F}_\bullet if the vector space and the filtration are clear from the context and write simply e_j in such a case.

Note that we have the following monotonicity

$$e_{\min}(E, \mathcal{F}_\bullet) := e_{\dim E}(E, \mathcal{F}_\bullet) \leq \dots \leq e_1(E, \mathcal{F}_\bullet) =: e_{\max}(E, \mathcal{F}_\bullet).$$

In particular,

$$e_{\min}(E, \mathcal{F}_\bullet) = \inf \{t \in \mathbb{R} : \mathcal{F}_t E \neq E\} \quad \text{and} \quad e_{\max}(E, \mathcal{F}_\bullet) = \sup \{t \in \mathbb{R} : \mathcal{F}_t E \neq 0\}.$$

Following Boucksom and Chen, we define the *mass* of (E, \mathcal{F}_\bullet) as

$$\text{mass}(E, \mathcal{F}_\bullet) := \sum_{j=1}^{\dim E} e_j(E, \mathcal{F}_\bullet).$$

Remark 3.4. Once the functions associated to filtrations will have been defined, the mass of a filtration will be related to the integral of the corresponding function over Newton–Okounkov bodies.

Example 3.5 (Jumping numbers on homogeneous polynomials). Let $X = \mathbb{P}^2$ and $E = H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. We consider the filtration \mathcal{F}_\bullet on E introduced by a geometric valuation ν given by the order of vanishing ord_p at a fixed point $p \in \mathbb{P}^2$ as in Example 3.2. Then

$$e_{\min} = e_3 = 0, \quad e_2 = e_1 = e_{\max} = 1 \quad \text{and} \quad \text{mass} = 2.$$

3.2 Filtrations on graded algebras

The constructions from the previous part extend to the setting of graded \mathbb{C} -algebras.

Definition 3.6 (A filtration on a graded object). Let

$$E_\bullet = \bigoplus_{k \geq 0} E_k$$

be a graded \mathbb{C} -algebra with $E_0 = \mathbb{C}$ and $\dim E_k$ finite for all k . A family $\mathcal{F}_\bullet E_\bullet$ of subspaces of E_\bullet is a *filtration of the graded algebra* E_\bullet if $\mathcal{F}_\bullet E_k$ is a filtration on the vector space E_k for all k .

We say that \mathcal{F}_\bullet is *multiplicative* if for all $s, t \in \mathbb{R}$ and all m, n we have

$$(\mathcal{F}_t E_m) \cdot (\mathcal{F}_s E_n) \subset \mathcal{F}_{t+s} E_{m+n}.$$

Example 3.7 (A filtration given by a valuation). Let X be an irreducible projective variety. Let $E_\bullet = \bigoplus_{k \geq 0} E_k T^k \subset \mathbb{C}(X)[T]$ be a graded subalgebra which is connected (i.e. $E_0 = \mathbb{C}$) and locally finite (that is, $\dim E_k < \infty$ for all k).

Let ν be a geometric valuation on $\mathbb{C}(X)$ i.e. a valuation defined by the order of vanishing along a subscheme Z in X . Since

$$\nu(f_1 \cdot f_2) = \nu(f_1) + \nu(f_2) ,$$

the expression

$$\mathcal{F}_t E_k = \{f \in E_k : \nu(f) \geq t\}$$

defines a multiplicative filtration.

Definition 3.8 (Linearly bounded filtrations). In the setup of Definition 3.6, we say that the filtration $\mathcal{F}_\bullet E_\bullet$ is *linearly left bounded*, if there exists a constant $C > 0$ such that for all k we have

$$e_{\min}(E_k, \mathcal{F}_\bullet) \geq -C \cdot k.$$

Similarly, $\mathcal{F}_\bullet E_\bullet$ is *linearly right bounded*, if

$$e_{\max}(E_k, \mathcal{F}_\bullet) \leq C \cdot k$$

for a fixed constant $C > 0$ and all k .

We can generalize jumping numbers to the graded setting.

Definition 3.9 (Asymptotic jumping numbers). With notation as in Definition 3.6 we set

$$e_{\min}(E_\bullet, \mathcal{F}_\bullet) := \liminf \frac{1}{k} e_{\min}(E_k, \mathcal{F}_\bullet) \quad \text{and} \quad e_{\max}(E_\bullet, \mathcal{F}_\bullet) := \limsup \frac{1}{k} e_{\max}(E_k, \mathcal{F}_\bullet).$$

Note that a filtration $\mathcal{F}_\bullet E_\bullet$ of the graded \mathbb{C} -algebra E_\bullet is linearly left bounded if and only if $e_{\min}(E_\bullet, \mathcal{F}_\bullet) > -\infty$ and similarly, it is linearly right bounded if and only if $e_{\max}(E_\bullet, \mathcal{F}_\bullet) < \infty$.

Proposition 3.10 (Filtration on a graded linear series). *Let X be an irreducible normal projective variety of dimension n , D a Cartier divisor on X and V_\bullet a graded linear series defined by D . Furthermore let Z be a subvariety in X , $\nu = \text{ord}_Z$ be the geometric valuation defined by Z , and let $\mathcal{F}_\bullet V_\bullet$ be the filtration given by ν as in Example 3.7. Then \mathcal{F}_\bullet is linearly left and right bounded.*

Proof. The valuation ord_Z is left bounded as $\text{ord}_Z(s) \geq 0$ for all $s \neq 0$, hence also

$$e_{\min}(V_k, \mathcal{F}_\bullet) \geq 0 \quad \text{for all } k.$$

For the right boundedness we claim that there exists a positive constant C such that

$$\max \{ \text{ord}_Z(s) : s \in V_k \} \leq C \cdot k$$

for all k . It is enough to prove this claim for the complete linear series $V_k = H^0(X, kD)$. To this end let $\pi : Y \rightarrow X$ be the blowing up along Z . There exists a unique irreducible component E of the exceptional locus of π mapping surjectively onto Z . For this component we have

$$\text{ord}_Z(s) = \text{ord}_E(\pi^* s) \quad \text{for all } s \in H^0(X, kD).$$

Let H be an ample line bundle on Y . There exists $C > 0$ such that

$$(\pi^* D - C E) \cdot H^{n-1} < 0.$$

This implies that $\text{ord}_Z(s) = \text{ord}_E(\pi^* s) \leq C \cdot k$ for all $s \in H^0(X, kD)$. Thus we have

$$e_{\max}(H^0(X, kD), \mathcal{F}_{\bullet}) = \max \{ \text{ord}_Z s : s \in H^0(X, kD) \} \leq C \cdot k.$$

□

Remark 3.11. In the setting of the above Proposition the asymptotic jumping number $e_{\min}(V_{\bullet}, \mathcal{F}_{\bullet})$ recovers the asymptotic order of vanishing $\text{ord}_Z(\|D\|)$ as defined in [5, Definition 2.2].

Example 3.12 (Asymptotic order of vanishing). Let X be a normal projective variety and V_{\bullet} a graded linear series on X . For a geometric valuation ν we define a filtration \mathcal{F}_{\bullet} on V_{\bullet} as in Proposition 3.10 and we set

$$\nu(V_k) := \min \{ \nu(s) : s \in V_k \setminus \{0\} \}.$$

Then

$$e_{\min}(V_k, \mathcal{F}_{\bullet}) = \nu(V_k)$$

and

$$e_{\min}(V_{\bullet}, \mathcal{F}_{\bullet}) = \lim_k \frac{1}{k} \nu(V_k) = \inf_k \frac{1}{k} \nu(V_k)$$

is the asymptotic order of vanishing along the center of ν . The fact that we can write \inf and \lim instead of \limsup is accounted for by the subadditivity of the sequence $\nu(V_k)$:

$$\nu(V_{k+m}) \leq \nu(V_k) + \nu(V_m)$$

as explained in [5, Lemma 2.1] and Fekete's Lemma [6].

The number e_{\max} behaves similarly under mild additional assumption.

Lemma 3.13 (e_{\max} for graded linear series). *Let V_{\bullet} be a graded linear series such that $V_k \neq 0$ for all k . Then*

$$e_{\max}(V_{\bullet}, \mathcal{F}_{\bullet}) = \lim_k \frac{1}{k} e_{\max}(V_k, \mathcal{F}_{\bullet}) = \sup_k \frac{1}{k} e_{\max}(V_k, \mathcal{F}_{\bullet})$$

for an arbitrary filtration on V_{\bullet} .

Proof. This follows by the superadditivity of the sequence $\{e_{\max}(V_k, \mathcal{F}_{\bullet})\}$ and again Fekete's Lemma, see also [1, Lemma 1.4]. □

Corollary 3.14 (Jumping numbers of Veronese algebra). *Let X be a normal projective variety and V_{\bullet} a graded linear series. Fixing a positive integer m , the Veronese algebra $V_{m\bullet}$ is a graded linear series as well. For a filtration \mathcal{F}_{\bullet} defined on V_{\bullet} by a geometric valuation μ on X and the corresponding filtration $\mathcal{F}_{m\bullet}$ on $V_{m\bullet}$, we have*

$$e_{\min}(V_{m\bullet}, \mathcal{F}_{m\bullet}) = m e_{\min}(V_{\bullet}, \mathcal{F}_{\bullet}) \quad \text{and} \quad e_{\max}(V_{m\bullet}, \mathcal{F}_{m\bullet}) = m e_{\max}(V_{\bullet}, \mathcal{F}_{\bullet}).$$

Proof. It follows from Example 3.12 and Lemma 3.13 that e_{\min} and e_{\max} scale well for graded subalgebras. □

We get the following characterization of the maximal jumping number in case of a complete linear series.

Remark 3.15 (Maximal jumping number of a complete linear series). Let X be a normal projective variety, Z an irreducible smooth subvariety of X . Let D be a Cartier divisor on X and $V_\bullet = R(X, D) = \bigoplus_{k \geq 0} H^0(X, kD)$ be the section ring of D . Moreover let $\pi : Y \rightarrow X$ be the normalized blowing up of Z with the exceptional divisor E . Then for $s \in H^0(X, kD)$ we have

$$\text{ord}_Z(s) = \text{ord}_E(\pi^*s) = \max \{m \in \mathbb{N} : \text{div}(\pi^*s) - mE \text{ is effective}\}.$$

Let \mathcal{F}_\bullet be the filtration on V_\bullet induced by the order of vanishing along Z . Then it follows from Example 3.12 that

$$e_{\max}(R(X, D), \mathcal{F}_\bullet) = \sup \frac{1}{k} \max \{ \text{ord}_Z(s) : s \in H^0(X, kD) \} = \\ \sup \{t \geq 0 : \pi^*L - tE \text{ is big}\} =: \mu_E(\pi^*D) =: \mu(D, Z).$$

Thus we see that e_{\max} is in this situation closely related to the geometry of the big cone on Y . Namely, it is the value of t at which the ray $\pi^*(L) - tE$ intersects the boundary of the big cone.

4 Functions on Okounkov bodies

Functions on Okounkov bodies have been studied by Boucksom and Chen [1] and Witt-Nyström [18]. As their approaches differ, we present here briefly both of them, keeping in mind that we will be interested later on in continuous functions on Okounkov bodies. As Proposition 4.9 shows, this is a quite delicate issue.

We fix for duration of this section a smooth projective variety X together with an admissible flag of subvarieties $Y_\bullet : X = Y_0 \supset \cdots \supset Y_n$.

4.1 Okounkov functions as concave envelopes

We begin with describing Witt-Nyström's construction. We recall first an auxiliary notion, see [17, Section 7].

Definition 4.1 (Closed concave envelope). Let $\Delta \subset \mathbb{R}^n$ be a compact, convex set, and let $f : \Delta \rightarrow \mathbb{R}$ be real valued function on Δ . The *closed concave envelope* f^c of f on Δ is defined by

$$f^c(x) = \inf \{g(x) | g \geq f, g \text{ concave and upper semi-continuous}\}.$$

The closed concave envelope of a bounded function f can be constructed as follows. Let H be the hypograph of f in $\Delta \times \mathbb{R}$, let H^c be the closed convex hull of H and define f^c to be the unique function on Δ having H^c as its hypograph, cf [17].

Remark 4.2. The function f^c is concave and upper semi-continuous (since its hypograph is closed). From its concavity it follows that f^c is continuous on the interior of Δ . Being concave and upper-semi-continuous, it is continuous along any line segment lying in Δ .

Witt-Nyström in [18] uses the above notion to define functions on Okounkov bodies. We now present his definition. From now on we work with a linearly bounded filtration \mathcal{F}_\bullet on V_\bullet (typically defined by a geometric valuation ν on the

function field $\mathbb{C}(X)$). In our approach we follow Witt-Nyström [18, Section 3] for an appropriate super-additive function.

We define the function $\widetilde{\varphi}_{\mathcal{F}_\bullet}$ at points $v \in \Delta_{Y_\bullet}(V_\bullet)$ which are normalized valuation vectors by

$$\widetilde{\varphi}_{\mathcal{F}_\bullet}(v) := \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} : \exists s \in \mathcal{F}_t V_k : \nu_{Y_\bullet}(s) = k \cdot v\}. \quad (6)$$

This limit exists because the sequence

$$a_k := \sup \{t \in \mathbb{R} : \exists s \in \mathcal{F}_t V_k : \nu_{Y_\bullet}(s) = k \cdot v\}$$

is superadditive, i.e. $a_k + a_l \leq a_{k+l}$ for all $k, l \geq 1$. Indeed, let $\varepsilon > 0$ be fixed. There exist sections

$$s_1 \in \mathcal{F}_{a_k - \varepsilon/2} V_k \quad \text{and} \quad s_2 \in \mathcal{F}_{a_l - \varepsilon/2} V_l$$

such that $\nu_{Y_\bullet}(s_1) = kv$ and $\nu_{Y_\bullet}(s_2) = lv$, so $(s_1 s_2) \in \mathcal{F}_{a_k + a_l - \varepsilon} V_{k+l}$ by the multiplicativity of the filtration and $\nu_{Y_\bullet}(s_1 s_2) = (k+l)v$. The existence of the limit now follows from Fekete's Lemma [6].

In points x which are not valuation vectors (in particular in such points that do not belong to $\Delta_{Y_\bullet}(V_\bullet)$) we set $\widetilde{\varphi}_{\mathcal{F}_\bullet}(x) := 0$. Thus the mapping $\widetilde{\varphi}_{\mathcal{F}_\bullet}$ is defined on the whole space \mathbb{R}^n . Now we are in a position to define the Okounkov function as in [18, Section 3].

Definition 4.3 (Okounkov function 1). Using the above notation, we set

$$\varphi_{\mathcal{F}_\bullet}(x) := \widetilde{\varphi}_{\mathcal{F}_\bullet}^c(x)$$

for all $x \in \Delta_{Y_\bullet}(V_\bullet)$. We call this concave function the *Okounkov function* associated to \mathcal{F}_\bullet .

If \mathcal{F}_\bullet is the filtration associated to a geometric valuation ν of the function field of X , then we will also use the notation φ_ν for $\varphi_{\mathcal{F}_\bullet}$.

Corollary 12.1.1 in [17] guarantees that this definition is the same as the one given in [18].

We observe now that taking concave envelope leaves the values of the underlying function $\widetilde{\varphi}_{\mathcal{F}_\bullet}$ in normalized valuation vectors untouched.

Lemma 4.4. *For an arbitrary normalized valuation vector v there is the equality*

$$\varphi_{\mathcal{F}_\bullet}(v) = \widetilde{\varphi}_{\mathcal{F}_\bullet}(v).$$

Proof. It suffices to show that the function $\widetilde{\varphi}_{\mathcal{F}_\bullet}$ is "concave" on the normalized valuation vectors. To this end, it suffices to show

$$\frac{1}{2} \widetilde{\varphi}_{\mathcal{F}_\bullet}(v) + \frac{1}{2} \widetilde{\varphi}_{\mathcal{F}_\bullet}(u) \leq \widetilde{\varphi}_{\mathcal{F}_\bullet}\left(\frac{1}{2}u + \frac{1}{2}v\right) \quad (7)$$

for all normalized valuation vectors u and v . Note that it follows from the proof of Lemma 2.7 that $\frac{1}{2}(u+v)$ is again a normalized valuation vector.

Let $\varepsilon > 0$ be fixed. It follows from the discussion right after (6) that the limit in (6) is actually a supremum. Hence there exist numbers $k, l \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, as well as sections $s_1 \in \mathcal{F}_{t_1} V_k$, $s_2 \in \mathcal{F}_{t_2} V_l$ such that $\nu_{Y_\bullet}(s_1) = ku$, $\nu_{Y_\bullet}(s_2) = kv$ and

$$\frac{t_1}{k} > \widetilde{\varphi}_{\mathcal{F}_\bullet}(u) - \varepsilon \quad \text{and} \quad \frac{t_2}{l} > \widetilde{\varphi}_{\mathcal{F}_\bullet}(v) - \varepsilon.$$

Then for $s = s_1^l \cdot s_2^k$ we have

$$s \in V_{2kl} \text{ and } \nu_{Y_\bullet}(s) = 2lk \left(\frac{1}{2}u + \frac{1}{2}v \right).$$

Moreover $s \in \mathcal{F}_{lt_1+kt_2} V_{2kl}$ by the multiplicity of \mathcal{F}_\bullet . Hence

$$\widetilde{\varphi_{\mathcal{F}_\bullet}} \left(\frac{1}{2}u + \frac{1}{2}v \right) \geq \frac{lt_1 + kt_2}{2lk} > \frac{1}{2} \widetilde{\varphi_{\mathcal{F}_\bullet}}(u) + \frac{1}{2} \widetilde{\varphi_{\mathcal{F}_\bullet}}(v) - \varepsilon$$

which implies (7). \square

Remark 4.5. Keeping the notation from above, let \mathcal{F}_\bullet be the valuation obtained from a geometric valuation ν , let D be a big divisor. Then

$$\inf_{\Delta_{Y_\bullet}(D)} \varphi_\nu \geq \nu(\|D\|),$$

where $\nu(\|D\|)$ denotes the asymptotic value of ν on D as defined in [5, Section 2].

Remark 4.6. It is an obvious but important consequence of Remark 4.2 that Okounkov functions are upper-semi-continuous.

Proof of Theorem 1.1, Part (i). According to [8, Proposition 3], all non-negative concave upper-semi-continuous functions are continuous on locally polyhedral subsets of \mathbb{R}^n . In particular, if $\Delta_{Y_\bullet}(V_\bullet)$ is a polytope (no matter whether it is rational or not), then all Okounkov non-negative Okounkov functions are automatically continuous on the whole of $\Delta_{Y_\bullet}(V_\bullet)$.

This latter statement includes in particular all Okounkov functions coming from geometric valuations. \square

4.2 Okounkov functions via graded linear series

Here we recall the construction by Boucksom and Chen. Let \mathcal{F}_\bullet be a multiplicative filtration on the graded linear series V_\bullet . Then, for any $t \in \mathbb{R}$, we can define a new graded linear series $V_\bullet^{(t)}$ via

$$V_k^{(t)} := \mathcal{F}_{tk} V_k \tag{8}$$

for all k . The Okounkov bodies $\Delta_{Y_\bullet}(V_\bullet^{(t)})$ form a non-increasing family of compact convex subsets of $\Delta_{Y_\bullet}(V_\bullet)$ and they have been used by Boucksom and Chen [1, Definition 1.8] in order to define functions on Okounkov bodies.

Definition 4.7 (Okounkov function 2). With notation as above, put

$$\psi_{\mathcal{F}_\bullet}(x) \stackrel{\text{def}}{=} \sup \left\{ t \in \mathbb{R} : x \in \Delta_{Y_\bullet}(V_\bullet^{(t)}) \right\}.$$

for all $x \in \Delta_{Y_\bullet}(V_\bullet)$ and call this function also the *Okounkov function* associated to \mathcal{F}_\bullet .

The following lemma states that these two definitions are equivalent.

Lemma 4.8. *The definitions 4.3 and 4.7 are equivalent on $\Delta_{Y_\bullet}(V_\bullet)$, i.e.*

$$\varphi_{\mathcal{F}_\bullet}(x) = \psi_{\mathcal{F}_\bullet}(x)$$

for all $x \in \Delta_{Y_\bullet}(V_\bullet)$.

Proof. In what follows, we will denote the closed convex closure of a subset of $S \subset \mathbb{R}^n$ by $\text{clconv}(S)$. Consider the set

$$H_1 = \{(x, y) | x \text{ a normalised valuation vector}, \exists v \text{ s.t. } \nu(v) = x, \text{val}(v) \geq y\} \subseteq \Delta \times \mathbb{R}.$$

Note that by definition

$$\Delta_t \stackrel{\text{def}}{=} \text{closed convex hull} (H_1 \cap \{\Delta \times t\}).$$

In particular, if we consider the set $H_2 \subset \Delta \times \mathbb{R}$ defined by

$$H_2 \cap \{\Delta \times t\} = \text{closed convex hull}(H_1 \cap \{\Delta \times t\})$$

then we have that

$$\psi_{\mathcal{F}_\bullet}(x) = \sup \{t \mid (x, t) \in H_2\}.$$

Observe that $H_1 \subset H_2 \subset \text{clconv}(H_1)$.

Let H_3 be the hypograph of $\psi_{\mathcal{F}_\bullet}$: we then have that $H_2 \subset H_3 \subset \text{cl}(H_2)$. Moreover, H_3 , as the hypograph of an upper semi-continuous concave function is automatically closed and convex, so $H_3 = \text{cl}(H_2)$, and this closure is a convex set. In particular, we have that $H_3 = \text{cl}(H_2) = \text{clconv}(H_1)$. Let H be the hypograph of $\widetilde{\varphi_{\mathcal{F}_\bullet}}$, so that $\text{clconv}(H)$ is the hypograph of $\varphi_{\mathcal{F}_\bullet}$. It is immediate from the definition that

$$H_1 \subset H \subset \text{cl}(H_1)$$

and hence $\text{clconv}(H) = \text{clconv}(H_1) = H_3$. The hypograph $\varphi_{\mathcal{F}_\bullet}$ is therefore equal to H_3 , which is the hypograph of $\psi_{\mathcal{F}_\bullet}$. These two functions are therefore equal. \square

4.3 An example of a non-continuous Okounkov function

Concave envelopes are in general only upper semicontinuous on the boundary. In the absence of good geometric properties of $\Delta_{Y_\bullet}(V_\bullet)$ (cf. Theorem 1.1 (i)), there is no guarantee that Okounkov functions defined on $\Delta_{Y_\bullet}(V_\bullet)$ will be continuous. Here we show by example that such a situation indeed can occur.

First, the following Proposition gives a sufficient condition for non-continuous behavior of an Okounkov function. After its proof we present an example where the circumstances described do happen.

Proposition 4.9 (A non-continuity criterion). *Let X be a variety, $Y_\bullet : X = Y_0 \supset Y_1 \supset \dots \supset Y_n$ a flag on X and D a divisor on X . Let $\Delta_{Y_\bullet}(D)$ be the Okounkov body of D with respect to this flag and let p be a point in the boundary of $\Delta(D)$ such that $p = \nu(s)$, where s is a section in $H^0(X, D)$ defining a reduced irreducible divisor Y and ν_{Y_\bullet} is the multivaluation associated to the flag Y_\bullet . Let v be the valuation associated to Y , i.e. $v = \text{ord}_Y$.*

If $\Delta_{Y_\bullet}(D)$ is not locally a cone around p , then the Okounkov function φ_v associated to the valuation v is not continuous at the point p .

Proof. Let us consider the Okounkov bodies $\Delta_t(D)$ associated to the filtration given by the valuation v . We have that $\Delta_t(D) = t\nu_{Y_\bullet}(s) + (1-t)\Delta_{Y_\bullet}(D)$ for $t \in [0, 1]$ and $\Delta_t(D) = \emptyset$ if $t > 1$. In other words, if $t \in [0, 1]$ then $\Delta_t(D)$ is produced from $\Delta_{Y_\bullet}(D)$ by performing on $\Delta_{Y_\bullet}(D)$ a homothety of ratio $(1-t)$ centered at the point $p = \nu_{Y_\bullet}(s)$.

In particular, $p \in \Delta_t(D)$ for all $t \in [0, 1]$ and hence

$$\varphi_v(p) = \sup\{t : p \in \Delta_t(D)\} = 1.$$

Since $\Delta_{Y_\bullet}(D)$ is not locally a cone around p we can find a sequence of points p_i contained in the boundary $\partial\Delta_{Y_\bullet}(D)$ such that

1. $\lim_{i \rightarrow \infty} p_i = p$;
2. For all integers i and $t > 0$ we have that $p + (1+t)(p_i - p) \notin \Delta_{Y_\bullet}(D)$.

In other words, the line passing through p and p_i leaves the Okounkov body exactly $\Delta_{Y_\bullet}(D)$ at the point p_i . In particular, it follows that $p_i \notin \Delta_t(D)$ for any $t > 0$ so that $\varphi_v(p_i) = 0$. It follows that φ_v is not continuous at the point p . This completes the proof of Proposition 4.9. \square

We will now produce a threefold X along with an admissible flag $X = Y_0 \supset Y_1 \supset Y_2 \supset Y_3$, a divisor D on X and a section s of D , such that $\nu(s)$ lies in the round part of the boundary of $\Delta(D)$.

Our example comes from [12], which is in turn heavily based on earlier work of Cutkosky [2]. The first part of the discussion is taken from [12] almost verbatim.

In [2], Cutkosky constructs a quartic surface $S \subseteq \mathbb{P}^3$ whose Néron-Severi space $N^1(S)$ is isomorphic to \mathbb{R}^3 with the lattice \mathbb{Z}^3 and the intersection form $q(x, y, z) = 4x^2 - 4y^2 - 4z^2$. He shows that

1. The divisor class $(1, 0, 0)$ on S corresponds to a very ample divisor class $[L]$ and the projective embedding corresponding to L realizes S as a quartic surface in \mathbb{P}^3 .
2. The nef and effective cones of S coincide, and are given by the conditions

$$v^2 \geq 0, \quad ([L] \cdot v) > 0.$$

Now, take the nef class $\alpha \stackrel{\text{def}}{=} (1, 1, 0) \in N^1(S)$, and let C be a curve with class $[C] = \alpha$. We note that since the effective cone of S has no polyhedral part, any curve C on S such that $C^2 = 0$ and $\frac{1}{k}[C]$ is not integral for any $k > 1$, is automatically irreducible. In this case, all members of the linear series of C are irreducible.

Since $C^2 = 0$, Riemann-Roch implies that $\chi(C) = 2$ hence $h^0(C) + h^2(C) = h^0(C) + h^0(-C) \geq 2$. As $(L \cdot (-C)) = -4$, we know that $h^0(-C) = 0$ and it follows that $h^0(C) \geq 2$. There is therefore a pencil of curves on S with the class α , no two different elements of this pencil meet because $\alpha^2 = 0$ and all members of the pencil are irreducible. This pencil is hence base point free and its general element is smooth by Bertini theorem. A general element $C \subset \mathbb{P}^3$ of this pencil is then a smooth elliptic curve of degree 4.

Let X to be the blow-up of \mathbb{P}^3 along the curve C . We denote by $Y_1 \subset X$ the proper transform of S in X . We note that Y_1 is isomorphic under projection to S .

We now choose a sufficiently positive ample divisor D on X , such that $D|_{Y_1}$ and $Y_1|_{Y_1}$ are independent in the Picard group of Y_1 . Moreover we can assume that all of the following divisors are ample:

$$D, \quad D - Y_1 - K_X, \quad D - Y_1, \quad D - 2Y_1 - K_X. \quad (9)$$

Furthermore, we choose a curve C' on $Y_1 = S$ such that $[C']$ is a primitive integral member of the boundary of $\text{Eff}(Y_1)$. (The class $[C']$ is effective by the Riemann-Roch argument given above). Moreover, we assume that C' is not contained in the image of the restriction map from $\text{Pic}(X)$ to $\text{Pic}(Y_1)$. We note that this implies that

$$[D|_{Y_1}], [Y_1|_{Y_1}] \text{ and } [C] \text{ are independent in } NS(Y_1). \quad (10)$$

Finally, we pick Y_2 to be a smooth curve contained in the class $D|_{Y_1} - C'$ and pick Y_3 to be a general point on Y_2 .

Proof of Theorem 1.1, (ii). With X, Y_1, Y_2, Y_3 and D as above, we now show that there is a reduced and irreducible divisor Z on X , linearly equivalent to D , such that close to the point $\nu(Z) \in \Delta(D)$ the set $\Delta(D)$ is not locally a cone. Here ν denotes the 3-valuation determined by the flag Y_\bullet .

Since D and $D - Y_1 - K_X$ are both ample by (9), the restriction map on global sections $H^0(D) \rightarrow H^0(D|_{Y_1})$ is surjective, and indeed so is $H^0(kD) \rightarrow H^0(kD|_{Y_1})$ for any k .

We can therefore choose a section of D determining a divisor Z not vanishing along Y_1 and such that $Z|_{Y_1} = Y_2 \cup C'$. By generality of Y_3 we then have

$$\nu(Z) = (0, 1, 0) .$$

Let us show now that the divisor Z is reduced and irreducible. If not then we can write Z as a sum of non-zero effective divisors

$$Z = Z' + Z'' .$$

Then $Z'|_{Y_1}$ and $Z''|_{Y_1}$ are non-zero effective divisors and $(Z' + Z'')|_{Y_1} = C' + Y_2$, where C' and Y_2 are both irreducible. Without loss of generality $Z'|_{Y_1} = C'$, but this contradicts our assumption that C' is not a restriction of a divisor on X .

Let us now show that $\Delta(D)$ is not locally a cone at $(0, 1, 0)$. We consider

$$\Delta'(D) = \{(a, b, c) | 0 \leq a \leq 1, (a, b, c) \in \Delta(D)\}$$

i.e. we consider a part of $\Delta(D)$ with a sufficiently small. From (9) it follows that for any k and any $a \in [0, 1]$ such that $ka \in \mathbb{N}$ the mapping

$$H^0(k(D - aY_1)) \rightarrow H^0(k(D - aY_1)|_{Y_1})$$

is surjective. It follows that for any $a \in [0, 1]$

$$\Delta(D) \cap \{(a, -, -)\} = \{(a, b, c) | (b, c) \in \Delta((D - aY_1)|_{Y_1})\}.$$

In other words, the slice of the Okounkov body $\Delta(D)$ with the plane $(a, -, -)$ is just the Okounkov body of $(D - aY_1)|_{Y_1}$ on Y_1 .

As Y_1 is a surface with no negative curves, the description of its Okounkov bodies given in [15, Theorem 6.4] is then very simply

$$\Delta(D - aY_1)|_{Y_1} = \{(b, c) | (D - aY_1)|_{Y_1} - bY_2 \text{ effective}, 0 \leq c \leq (D - aY_1 - bY_2) \cdot Y_2\}$$

or in other words

$$\Delta'(D - aY_1) = \{(a, b, c) | 0 \leq a \leq 1, f_1 \geq 0, f_2 > 0, 0 \leq c \leq f_3\} ,$$

where

$$\begin{aligned} f_1 &= (D|_{Y_1} - aY_1|_{Y_1} - bY_2)^2, \\ f_2 &= (D|_{Y_1} - aY_1|_{Y_1} - bY_2) \cdot L, \\ f_3 &= (D|_{Y_1} - aY_1|_{Y_1} - bY_2) \cdot Y_2. \end{aligned}$$

For simplicity, let us now consider the slice

$$\Delta''(D) = \{(a, b) | (a, b, 0) \in \Delta_\epsilon(D)\}$$

obtained by intersecting $\Delta'(D)$ and the plane $c = 0$. Alternatively, we can write

$$\Delta''(D) = \{(a, b) | 0 \leq a \leq 1, (D|_{Y_1} - aY_1|_{Y_1} - bY_2)^2 \geq 0 \text{ and } (D|_{Y_1} - aY_1|_{Y_1} - bY_2) \cdot L > 0\}.$$

It will be enough to show that $\Delta''(D)$ is not locally a cone around the point $(0, 1)$. Recall that any cone in \mathbb{R}^2 is either the whole of \mathbb{R}^2 or is bounded by two straight half-lines. $(0, 1)$ is not an interior point of $\Delta''(D)$ so the first possibility is excluded.

The set $\Delta''(D)$ is bounded by the following curves:

1. the x -axis,
2. the y -axis,
3. the line $x = \epsilon$,
4. the branch of the conic section defined by the equation

$$(D|_{Y_1} - aY_1|_{Y_1} - bY_2)^2 = 0.$$

passing through $(0, 1)$.

The point $(0, 1)$ lies at the intersection of the y -axis and the conic section defined by the equation $(D|_{Y_1} - aY_1|_{Y_1} - bY_2)^2 = 0$.

To establish that $\Delta''(D)$ is not locally a cone around b it will be enough to show that the conic section given by the equation $(D|_{Y_1} - aY_1|_{Y_1} - bY_2)^2 = 0$ does not contain a straight line. This conic section is the intersection in $\text{Pic}(Y_1)$ of the nef cone $x^2 = y^2 + z^2$ with the plane passing through the points $D|_{Y_1}$, $(D - Y_1)|_{Y_1}$ and $D|_{Y_1} - Y_2$. By (10) this plane does not pass through 0 so the resulting conic section is not a union of straight lines. This completes the proof of Theorem 1.1. \square

4.4 Examples

We devote this section to several examples where functions associated to various geometric valuations are determined explicitly. First we deal with the one-dimensional case, where Okounkov functions associated to complete linear series can be computed in general.

Example 4.10 (Okounkov function of a valuation on a curve). Let C be a smooth curve, V_\bullet a big graded linear system associated to a line bundle L of positive degree, and let

$$Y_\bullet : C \supset \{p\}$$

be a fixed flag.

a) Consider the filtration $\mathcal{F}_\bullet = \text{ord}_p$ on V_\bullet defined by the order of vanishing at the point p in the flag.

Let $x \in \Delta_{Y_\bullet}(V_\bullet)$ be arbitrary, and write it as a limit of normalized valuation vectors $x = \lim_{k \rightarrow \infty} \frac{\alpha_k}{k}$. Then

$$\begin{aligned} \varphi_{\text{ord}_p}(x) &= \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} : \exists s \in V_k : \text{ord}_p(s) \geq t \text{ and } \text{ord}_p(s) = k\alpha_k\} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \alpha_k \\ &= x . \end{aligned}$$

It turns out that in this case the Okounkov function is the identity.

b) Now consider the filtration $\mathcal{F}_\bullet = \text{ord}_q$ defined by the order of vanishing in a point q not in the flag. In this case, we take V_\bullet to be the complete graded linear series associated to the divisor L . At a point $x = \lim_{k \rightarrow \infty} \frac{\alpha_k}{k}$ as above, we have

$$\begin{aligned} \varphi_{\text{ord}_q}(x) &= \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} : \exists s \in V_k : \text{ord}_q(s) \geq t \text{ and } \text{ord}_p(s) = k\alpha_k\} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} (k \deg(L) - \alpha_k) \\ &= \deg(L) - x . \end{aligned}$$

Next, we move on the surfaces, where calculations become very difficult very soon. This is not surprising, since invariants of Okounkov functions on surfaces already carry deep geometric information (see [13]).

Example 4.11 (Okounkov function of a valuation on the projective plane). Set $X_0 = \mathbb{P}^2$, $D_0 = \mathcal{O}_{\mathbb{P}^2}(1)$, and let $P_0 \in \ell \subset X_0$ be a flag as in Example 2.4.

a). First, we handle the case $\nu = \text{ord}_{P_0}$. In the rational points $(a, b) \in \Delta(D_0)$ the Okounkov function φ^0 is then

$$\begin{aligned} \varphi^0(a, b) &= \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} : \exists s \in |kD_0| : \text{ord}_\ell(s) = ka, \text{ord}_{P_0}(s_1) = kb, \\ &\quad \text{ord}_{P_0}(s) \geq t\} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} k(a + b) \\ &= (a + b) , \end{aligned}$$

where s_1 is defined as in (1). As the Okounkov body $\Delta(D_0)$ is a polytope, φ^0 is continuous by Theorem 1.1, hence $\varphi^0(a, b) = a + b$ for all $(a, b) \in \Delta(D_0)$. We point out that using the definition of Boucksom and Chen, one can obtain the result without referring to the continuity of φ^0 .

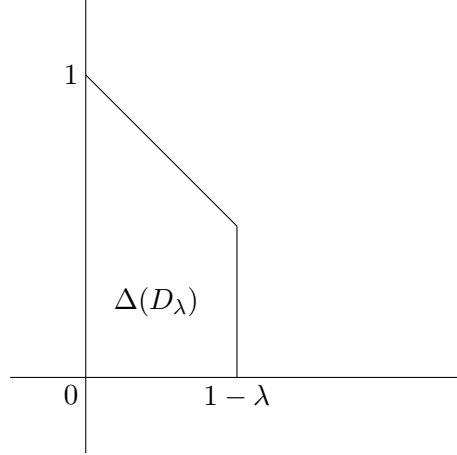
b) Now we consider $\nu = \text{ord}_{P_1}$ for a point P_1 not on the line ℓ . For the rational points $(a, b) \in \Delta(D_0)$ we have

$$\begin{aligned} \varphi^1(a, b) &= \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} : \exists s \in |kD_0| : \text{ord}_\ell(s) = ka, \text{ord}_{P_0}(s_1) = kb, \\ &\quad \text{ord}_{P_1}(s) \geq t\} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} k(1 - a) \\ &= 1 - a . \end{aligned}$$

Again, the same formula holds over the whole of $\Delta(D_0)$ by a continuity argument.

Note that the analogous calculations can be carried out on a projective space of arbitrary dimension.

Example 4.12 (Okounkov function on a blow up of the projective plane). Keeping the notation of the Example 4.11, let $f : X_1 = \text{Bl}_{P_1} X_0 \rightarrow \mathbb{P}^2$ be the blow up of the projective plane in a point P_1 not contained in the flag line ℓ with exceptional divisor E_1 . We work now with a \mathbb{Q} -divisor $D_\lambda = f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \lambda E_1$, for some fixed $\lambda \in [0, 1]$. A direct computation using [15, Theorem 6.2] gives that the Okounkov body has the shape



a). For the valuation $\nu = \text{ord}_{P_0}$, we get exactly as above

$$\varphi^0(a, b) = a + b.$$

b). For the valuation $\nu = \text{ord}_{P_2}$, where P_2 is a point in X_1 not on the exceptional divisor E_1 (hence P_2 can be considered also as a point on \mathbb{P}^2) and not on the line through P_0 and P_1 . We have now for $(a, b) \in \Delta(D_\lambda)$

$$\varphi^1(a, b) = \begin{cases} 1 - a & \text{for } a + b \leq 1 - \lambda \\ 2 - 2a - b - \lambda & \text{for } 1 - \lambda \leq a + b \leq 1 \end{cases}$$

This can be seen as follows. $\varphi^1(a, b)$ is the maximal order of vanishing at P_2 among all \mathbb{Q} -sections vanishing

- a) along ℓ to order a ;
- b) in P_1 to order λ ;
- c) in P_0 to order b after dividing by the equation of ℓ in power a and after restricting to ℓ .

Condition a) "costs" aH , so we are left with $(1-a)H - \lambda E_1$ to take care of conditions b) and c). If $b \leq 1 - a - \lambda$, then we take a line through the points P_2 and P_1 with multiplicity λ and the line through P_2 and P_0 with multiplicity $1 - a - \lambda$. Their union has multiplicity $\lambda + (1 - a - \lambda) = 1 - a$ at P_2 and satisfies b) and c). Moreover, there is no \mathbb{Q} -divisor equivalent to $(1-a)H - \lambda E_1$ with higher multiplicity at P_2 , which follows easily from Bézout's theorem intersecting with both lines.

The argument in the remaining case $b > 1 - a - \lambda$ is similar. We want to split the divisor so that it produces a high vanishing order towards condition c) first and then, after arriving to the threshold

$$b' = 1 - a' - \lambda', \quad (11)$$

we take again the union of two lines as above. Thus, we start with the conic through P_1 and P_2 tangent to ℓ at P_0 . We take this conic with multiplicity α subject to condition that

$$b - 2\alpha = 1 - a - 2\alpha - (\lambda - \alpha),$$

which means that the divisor $(1 - a - 2\alpha)H - (\lambda - \alpha)E_1$ satisfies (11) with $b' = b - 2\alpha$, $a' = a + 2\alpha$ and $\lambda' = \lambda - \alpha$. The constructed \mathbb{Q} -divisor, consisting of the conic and two lines has then multiplicity

$$a + b + \lambda - 1 + (1 - a - 2(a + b + \lambda - 1)) = 2 - 2a - b - \lambda.$$

Bézout's theorem shows then that there is no divisor of higher multiplicity.

4.5 Invariants of Okounkov functions

We treat various properties of Okounkov functions.

Given a linearly bounded filtration \mathcal{F}_\bullet on a graded linear series V_\bullet , we can restrict it to $\mathcal{F}_{m\bullet}$ on the Veronese subseries

$$V_{m\bullet} := \bigoplus_{k=1}^{\infty} V_{mk}$$

for $m \geq 1$. The index m in $\mathcal{F}_{m\bullet}$ helps us to keep track to which graded linear series the valuation is applied in the given moment. The corresponding Okounkov bodies scale well by [15, Proposition 4.1]

$$\Delta_{Y_\bullet}(V_{m\bullet}) = m\Delta_{Y_\bullet}(V_\bullet),$$

so that it makes sense to compare the corresponding Okounkov functions. It turns out that they scale as well.

Theorem 4.13 (Veronese homogeneity of Okounkov functions). *Let X be an irreducible projective variety and let \mathcal{F}_\bullet be a linearly bounded valuation on the graded linear series V_\bullet . Then*

$$\varphi_{\mathcal{F}_{m\bullet}}(mx) = m \cdot \varphi_{\mathcal{F}_\bullet}(x) \quad (12)$$

for all $x \in \Delta_{Y_\bullet}(V_\bullet)$.

Proof. To begin with let $x \in \Delta_{Y_\bullet}(V_\bullet)$ be a normalized valuation vector. Then

$$\begin{aligned} \varphi_{\mathcal{F}_{m\bullet}}(mx) &= \sup \{t \in \mathbb{R} : \exists s \in \mathcal{F}_{mt}V_k(mL) : \nu_{Y_\bullet}(s) = mx\} \\ &= \sup \{t \in \mathbb{R} : \exists s \in \mathcal{F}_{mt}V_{mk}(L) : \nu_{Y_\bullet}(s) = mx\} \\ &= m \sup \{t \in \mathbb{R} : \exists s \in \mathcal{F}_tV_k(L) : \nu_{Y_\bullet}(s) = x\} = m \varphi_{\mathcal{F}_\bullet}(x). \end{aligned}$$

The equality of both functions follows then from the density statement 2.7 (1.) and the fact that the closed concave envelope is unique. \square

Using the above result we show that working with an appropriate flag, the Okounkov function can be recovered out of its values on the boundary of $\Delta_{Y_\bullet}(V_\bullet)$. More precisely, we establish the following fact.

Theorem 4.14 (Reading Okounkov function out of boundary). *Assume that V_\bullet is a graded linear series associated to the line bundle L such that there is an irreducible divisor $Y_1 \in |L|$. We take a flag Y_\bullet whose divisorial part is Y_1 . Let \mathcal{F}_\bullet be a filtration on V_\bullet defined by a geometric valuation ν . Then for $x = (x_1, \dots, x_n) \in \Delta_{Y_\bullet}(V_\bullet)$ we have*

$$\varphi_{\mathcal{F}_\bullet}(x_1, \dots, x_n) = (1 - x_1)\varphi_{\mathcal{F}_\bullet}\left(0, \frac{x_2}{1 - x_1}, \dots, \frac{x_n}{1 - x_n}\right) + x_1 \cdot \nu(Y_1). \quad (13)$$

Proof. It suffices to establish (13) in case $x = v$ is a normalized valuation vector. For m large enough and divisible all coordinates mx_1, \dots, mx_n are integers and we have by (12)

$$\varphi_{\mathcal{F}_\bullet}(x) = \frac{1}{m}\varphi_{\mathcal{F}_{m\bullet}}(mx) = \frac{1}{m} \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t : \exists s \in V_{mk} : \nu(s) \geq t \text{ and } \nu_{Y_\bullet}(s) = mkx\}. \quad (14)$$

A section s with $\nu_{Y_\bullet}(s) = mkx$ can be written as $s = s' \cdot s_1^{mkx_1}$, where $s_1 \in H^0(L)$ is the section defining Y_1 and we have

$$\nu(s) = \nu(s') + mkx_1 \cdot \nu(s_1),$$

since ν is a geometric valuation. For the Okounkov valuation ν_{Y_\bullet} we have

$$\nu_{Y_\bullet}(s_1^{mkx_1}) = (mkx_1, 0, \dots, 0) \text{ and } \nu_{Y_\bullet}(s') = (0, mkx_2, \dots, mkx_n) =: x'.$$

Note that s' is a section in $V_{(1-x_1)mk}$. Thus, continuing (14) we establish

$$\begin{aligned} \varphi_{\mathcal{F}_\bullet}(x) &= \frac{1}{m} \lim_{k \rightarrow \infty} \frac{1}{k} \left[mkx_1 \nu(s_1) + \sup \{t : \exists s' \in V_{(1-x_1)mk} : \nu(s') \geq t \text{ and } \right. \\ &\quad \left. \nu_{Y_\bullet}(s') = mkx'\} \right]. \end{aligned} \quad (15)$$

With $x'' := \frac{1}{1-x_1} \cdot x'$ we have

$$mkx' = (1 - x_1)mk \cdot x''$$

and thus continuing (15) we have

$$\begin{aligned} \varphi_{\mathcal{F}_\bullet}(x) &= x_1 \nu(s_1) + (1 - x_1) \frac{1}{m(1 - x_1)} \times \\ &= \times \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t : \exists s' \in V_{(1-x_1)mk} : \nu(s') \geq t \\ &\quad \text{and } \nu_{Y_\bullet}(s') = (1 - x_1)mkx''\} \\ &= x_1 \nu(s_1) + (1 - x_1) \frac{1}{m(1 - x_1)} \varphi_{\mathcal{F}_{m(1-x_1)\bullet}}(m(1 - x_1)x'') \\ &= x_1 \nu(s_1) + (1 - x_1) \varphi_{\mathcal{F}_\bullet}(x''). \end{aligned}$$

□

Remark 4.15. Repeated applications of Theorem 4.14 reduce the computation of $\varphi_{\mathcal{F}_\bullet}$ to the situation where we consider only those global sections of L that vanish at the point Y_n . If the restriction map

$$H^0(X, \mathcal{O}_X(mL)) \longrightarrow H^0(Y_{n-2}, \mathcal{O}_{Y_{n-1}}(mL))$$

is surjective for $m \gg 0$, then this amounts to a calculation on the curve Y_{n-1} . Consequently, the computation of $\varphi_{\mathcal{F}_\bullet}$ for very ample divisors can be essentially reduced to the curve case.

At last we check that the functions φ_ν are continuous when considered as functions on the interior of the global Okounkov body of X .

Proposition 4.16. *Let X be an irreducible projective variety, ν a geometric valuation of $\mathbb{C}(X)$, $\varphi_\nu : \Delta_{Y_\bullet}(X) \rightarrow \mathbb{R}_{\geq 0}$ the associated Okounkov function. Then φ_ν is continuous on the open subset*

$$U \stackrel{\text{def}}{=} \bigcup_{\alpha \in \text{Big}(X)} \Delta_{Y_\bullet}^\circ(\alpha) \subseteq N^1(X)_{\mathbb{R}}.$$

Proof. Let D_1, \dots, D_ρ be integral divisors on X whose numerical equivalence classes form a \mathbb{Z} -basis of $N^1(X)_{\mathbb{R}}$; assume in addition that every effective divisor on X is a non-negative integral linear combination of the D_i 's up to numerical equivalence. This can be arranged by [15, p.30.]. For an element $\overline{m} \in \mathbb{N}^\rho$, we set as usual

$$\overline{m} \cdot \overline{D} \stackrel{\text{def}}{=} \sum_{i=1}^{\rho} m_i D_i.$$

The multigraded semigroup of X (with respect to the choices of the D_i 's and an admissible flag) is

$$\Gamma_{Y_\bullet}(X) = \{(\overline{m}, \nu_{Y_\bullet}(s)) \mid 0 \neq s \in H^0(X, \mathcal{O}_X(\overline{m} \cdot \overline{D}))\} \subseteq \mathbb{N}^{n+\rho}.$$

The global Okounkov body of X is then the closure of the convex hull of the set of normalized multigraded valuation vectors

$$\bigcup_{\overline{q} \in \mathbb{Q}_{\geq 0}^\rho} \bigcup_{k \in \mathbb{N}, k\overline{q} \in \mathbb{N}^\rho} \left\{ \left(\overline{q}, \frac{1}{k} \nu_{Y_\bullet}(s) \right) : 0 \neq s \in H^0(X, \mathcal{O}_X(k \cdot \overline{q} \cdot \overline{D})) \right\} \subseteq \mathbb{R}^{n+\rho}.$$

If (\overline{q}, α) is such a vector, then we define

$$\widetilde{\varphi}_\nu(\overline{q}, \alpha) \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_t H^0(X, \mathcal{O}_X(k \cdot \overline{q} \cdot \overline{D})) : \nu_{Y_\bullet}(s) = k \cdot \alpha \}.$$

For all other points of $\mathbb{R}^{n+\rho}$ we set $\widetilde{\varphi}_\nu$ to be equal to zero. The concave transform of $\widetilde{\varphi}_\nu$ is then a continuous function on $\Delta_{Y_\bullet}(X)$, which agrees over all classes $\xi \in N^1(X)_{\mathbb{Q}}$ with φ_ν defined on the Okounkov body $\Delta_{Y_\bullet}(\xi)$. This proves the claim. \square

5 Integrals of Okounkov functions

In this section we point out a new way of constructing invariants of numerical equivalence classes of Cartier divisors via integrating functions on Okounkov bodies. Let X be an irreducible projective variety of dimension n as so far, Y_\bullet an admissible flag.

Definition 5.1. Let V_\bullet be a big graded linear series, ν a geometric valuation of $\mathbb{C}(X)$. We set

$$I(V_\bullet; \nu) \stackrel{\text{def}}{=} \int_{\Delta_{Y_\bullet}(V_\bullet)} \varphi_\nu .$$

As usual, we write $I(D; \nu)$, whenever V_\bullet is the complete graded linear series associated to a Cartier divisor D on X .

Remark 5.2. The function φ_ν is a bounded upper-semicontinuous concave function on the compact subset $\Delta_{Y_\bullet}(V_\bullet)$, therefore it is Lebesgue integrable. Being non-negative as well, its integral is non-negative, and so $0 \leq I(D; \nu) < \infty$.

It follows from results of [1] that $I(D; \nu)$ is in fact independent of the flag Y_\bullet as the notation suggests.

Proposition 5.3. *With notation as above,*

$$I(D; \nu) \stackrel{\text{def}}{=} \text{vol}_{\mathbb{R}^n} \left(\hat{\Delta}(V_\bullet, F_\nu) \right) = \int_{t=0}^{+\infty} \text{vol}_{\mathbb{R}^n} \left(\Delta(V^{(t)}) \right) dt = \lim_{k \rightarrow +\infty} \frac{\text{mass}(V_k, F_\nu)}{k^{n+1}} .$$

Proof. This is the content of [1, Corollary 1.11]. Observe that the right-hand side expression is by its definition independent of the flag Y_\bullet . \square

Example 5.4. Let $f : X_1 \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 in a point P_1 with the exceptional divisor E_1 , as in Example 4.12. A divisor

$$D = \alpha f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \beta E_1 \text{ is big on } X_1 \text{ iff } \alpha > 0 \text{ and } \beta < \alpha.$$

For $\beta < 0$ we have

$$\Delta(\alpha f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \beta E_1) = \Delta(\alpha f^*(\mathcal{O}_{\mathbb{P}^2}(1))),$$

so it is enough to consider $0 \leq \beta < \alpha$. Furthermore, we can divide by α , see Remark 5.7. Then with $\lambda = \frac{\beta}{\alpha}$, it follows from Example 4.12 that

$$I(f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \lambda E_1, \text{ord}_P) = \frac{1}{3} - \frac{1}{2}\lambda^2 + \frac{1}{6}\lambda^3$$

for P as in case a). or b). of that example.

The next statement fits in well with the philosophy that asymptotic invariants of line bundles tend to respect numerical equivalence.

Proposition 5.5 (Numerical invariance of Okounkov functions). *Let v be a discrete valuation, D a big integral Cartier divisor on X . Then the function*

$$\varphi_v : \Delta_{Y_\bullet}(D) \longrightarrow \mathbb{R}$$

depends only on the numerical equivalence class of D .

Proof. Fix an arbitrary numerically trivial divisor P on X . First of all, as observed in [15, Proposition 4.1], Okounkov bodies are invariant with respect to numerical equivalence of divisors,

$$\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D + P) ,$$

whence the respective domains of the functions $\varphi_{D,\nu}$ and $\varphi_{D+P,\nu}$ agree.

Next, following the train of thought of the proof of [15, Proposition 4.1 (i)], we show that

$$\Delta(|\bullet(D+P)|^{(t)}) = \Delta(|\bullet D|^{(t)})$$

holds for every $t \in \mathbb{R}$.

We recall that $\Delta(|\bullet D|^{(t)})$ is the Newton–Okounkov body attached to the graded linear series

$$A_k \stackrel{\text{def}}{=} \{s \in H^0(X, \mathcal{O}_X(kD)) \mid v(s) \geq tk\} ,$$

while $\Delta(|\bullet(D+P)|^{(t)})$ is the convex body associated to the graded linear series

$$B_k \stackrel{\text{def}}{=} \{s' \in H^0(X, \mathcal{O}_X(k(D+P))) \mid v(s') \geq tk\} .$$

It follows from a Castelnuovo–Mumford regularity argument (see [14, Lemma 2.2.42]) that there exists a divisor B on X such that $B + lP$ is very ample for all $l \in \mathbb{Z}$. Let $a \gg 0$ be such that $|aD - B| \neq \emptyset$, and let $s \in H^0(X, \mathcal{O}_X(aD - B))$ be the section corresponding to an effective divisor. We write

$$(k+a)(D+P) \sim kD + (aD - B) + (B + (k+a)P) .$$

If we represent $B + (k+a)P$ by a section not going through the elements of Y_\bullet , then we obtain

$$A_k \cdot s \subseteq B_k ,$$

hence

$$\Gamma(A_k) + \nu(s) \subseteq \Gamma(B_k) .$$

By taking limits we obtain

$$\Delta_{Y_\bullet}(|\bullet D|^{(t)}) = \Delta_{Y_\bullet}(A_\bullet) \subseteq \Delta_{Y_\bullet}(B_\bullet) = \Delta_{Y_\bullet}(|\bullet(D+P)|^{(t)}) .$$

Replacing D by $D + p$ and P by $-P$ in the above argument yields the reverse inclusion. \square

Proposition 5.6. *Let X be an irreducible projective variety, ν a geometric valuation of its function field. Then both*

$$I(\cdot, \nu) : \text{Big}(X) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{and} \quad \frac{1}{\text{vol}_X(\cdot)} \cdot I(\cdot, \nu) : \text{Big}(X) \longrightarrow \mathbb{R}_{\geq 0}$$

are continuous functions.

Proof. The first claim is a consequence of Lebesgue’s dominated convergence theorem and the convexity properties of Okounkov functions. For the second claim, note that the volume function is continuous and non-zero on $\text{Big}(X)$. \square

Remark 5.7. Change of variables in the integral and homogeneity of Okounkov functions yield

$$I(mD; \nu) = m^{n+1} \cdot I(D; \nu) \quad \text{and} \quad \frac{1}{\text{vol}_X(mD)} \cdot I(mD, \nu) = m \cdot \frac{1}{\text{vol}_X(D)} \cdot I(D, \nu) .$$

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